Short Introduction to Derivatives and Integrals

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1 Derivative

The derivative is an operator which takes a function and yields the change rate (or slope) of the function for a specified variable.

The derivative operator with respect to variable x is symbolized by $\frac{d}{dx}$ (this is just a symbol, not a division and d is not a variable), and precedes the function upon which it operates. Thus the derivative of function f with respect to variable x is given by $\frac{df}{dx}$. If f is just a function of a single variable x, it is commonly written as $f(x)$, and its derivative with respect to x is also a function of x and is commonly written as $f'(x)$, where $f'(x) = \frac{df}{dx}$.

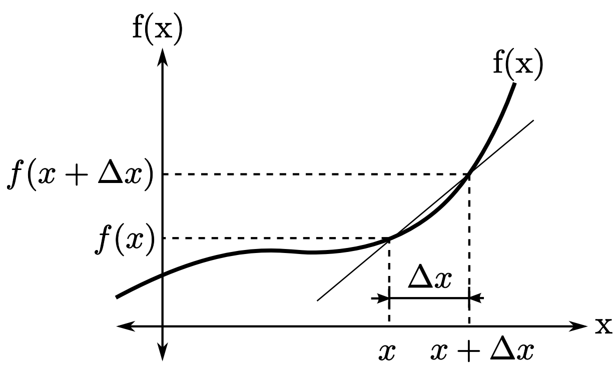

Figure 1: Approximation to the derivative of f at x.

To calculate the derivative of a function f for any value of x, consider the approximation for which the slope is calculated between two points (in (x, y) Cartesian coordinates) given by $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$ as shown in figure 1. Δx is a single variable, not the multiplication of Δ and x. The slope is given by the change in f divided by the change in x between the two points:

$$\frac{df}{dx} \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

To make the approximation exact we need to make Δx infinitely small such that the second point is infinitely close to the first point. Mathematically, this is given by:

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad (1)$$

This is the mathematical definition of the derivative.

Example 1:

Let $f(x) = x^2$. Find df/dx.

Using the definition we get:

$$\begin{aligned}
\frac{df}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + (\Delta x)^2 + 2x\Delta x - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 2x\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \Delta x + 2x \\
&= 2x
\end{aligned}$$

Using the definition becomes very difficult and impractical for any function other than low order polynomials. The best approach is to use properties and rules of the derivative operator.

Property: Linearity

The derivative is a linear operator. If c is a constant, and $f(x)$ and $g(x)$ are functions, then:

$$\frac{d}{dx}\left\{f(x) + c\, g(x)\right\} = \frac{d}{dx} f(x) + c\,\frac{d}{dx} g(x) \quad (2)$$

Formulas

$f(x)$	df/dx
x^n	nx^{n-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$

Rule 1

If $f(x) = g(x)h(x)$ then

$$\frac{df}{dx} = g'(x)h(x) + g(x)h'(x) \tag{3}$$

Rule 2

If $f(x) = g(h(x))$ then

$$\frac{df}{dx} = h'(x)g'(h(x)) \tag{4}$$

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These properties and rules should allow you to find the derivative of any function.

Example 2:

If $f(x) = x^3 + 2\sin(x)$, find $f'(x)$.

Applying (2), we can re-write $f'(x)$ into two simple expressions:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left\{x^3 + 2\sin(x)\right\} \\ &= \frac{d}{dx}x^3 + 2\frac{d}{dx}\sin(x) \end{aligned}$$

We can now solve the derivative using the formulas from the given table:

$$f'(x) = 3x^2 + 2\cos(x)$$

Example 3:

If $f(x) = \cos(x)x^2$, find $f'(x)$.

We can use (3). Let $g(x) = \cos(x)$ and $h(x) = x^2$ such that $f(x) = g(x)h(x)$. The derivatives of g and h can be found directly from the formulas in the table: $g'(x) = -\sin(x)$ and $h'(x) = 2x$, then

$$\begin{aligned} f'(x) &= g'(x)h(x) + g(x)h'(x) \\ &= -\sin(x)x^2 + 2\cos(x)x \end{aligned}$$

Example 4:

If $f(x) = e^{x^2}$, find $f'(x)$.

Let $g(x) = e^x$ and $h(x) = x^2$ such that $f(x) = g(h(x))$. Now we can use (4):

$$\begin{aligned} f'(x) &= h'(x)g'(h(x)) \\ &= 2xe^{x^2} \end{aligned}$$

Note that $g'(x) = e^x$ so $g'(h(x)) = e^{h(x)} = e^{x^2}$.

Example 5:

If $f(x) = 4\sin(x^2)x$, find $f'(x)$.

Write $f(x)$ as the product of two functions first and apply (3). Let $g(x) = \sin(x^2)$ and $h(x) = x$, then

$$\begin{aligned} f'(x) &= 2(g'(x)h(x) + g(x)h'(x)) \\ &= 2(g'(x)x + \sin(x)) \end{aligned}$$

Now, to find $g'(x)$, let $p(x) = \sin(x)$ and $q(x) = x^2$, and apply (4):

$$\begin{aligned} g'(x) &= q'(x)p'(q(x)) \\ &= 2x\cos(x^2) \end{aligned}$$

Then

$$f'(x) = 2(2x\cos(x^2) + \sin(x))$$

2 Integral

The integral is an operator which takes a function and yields the area under the curve described by the function between two values of the independent variable. The integral of a function $f(x)$ between a and b is symbolized by:

$$\int_a^b f(x)dx$$

$\int_a^b$ and dx are just symbols representing the operator and should not be taken as variables that multiply $f(x)$. This concept is illustrated in figure 2.

To calculate the integral of a function f between a and b, consider the approximation where the area under the curve is divided into rectangles of width

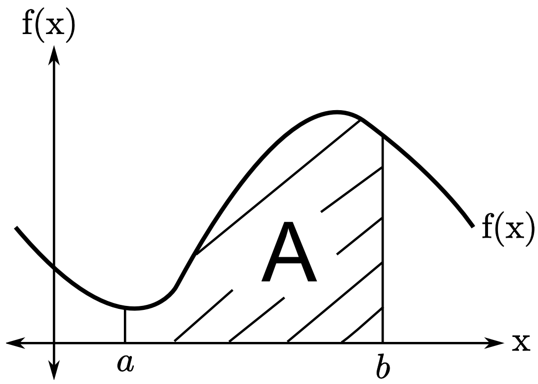

Figure 2: Graphical representation of the integral.

Δx and height equal to the value of f where the left edge of the rectangle sits. This is illustrated in figure 3. Then, the area under the curve between a and b is approximately the sum of the areas of the individual rectangles:

$$A \approx \sum_k f(x_k)\Delta x$$

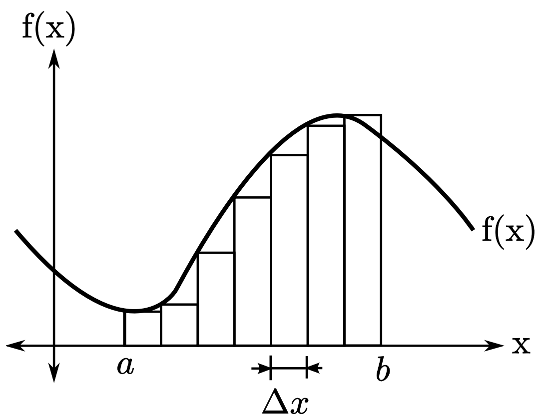

Figure 3: Approximation of the integral.

As you can see in figure 3, this is only an approximation. As Δx becomes smaller, the sum of the areas of the individual rectangles approaches the exact area under the curve. The exact area is the infinite sum of the areas of infinitely narrow rectangles.

To calculate the integral, consider first the concept of **anti-derivative** or **indefinite integral**, which is the inverse of the derivative: If the derivative of $F(x)$ is $f(x)$, then

$$\int f(x)dx = F(x) + c \tag{5}$$

Note the presence of an arbitrary constant c. It's there because no matter what the value of c is, given that the derivative of a constant is zero, it's always true that

$$\frac{d}{dx}\left\{F(x) + c\right\} = f(x) \tag{6}$$

Example 6:

If $f(x) = \cos(x)$, find $\int f(x)dx$.

The function we are looking for is such that its derivative is equal to $\cos(x)$. From the table of derivative formulas we know that $\frac{d}{dx}\sin(x) = \cos(x)$, then

$$\int f(x)dx = \int \cos(x)dx = \sin(x)$$

Example 7:

If $f(x) = x^3$, find $\int f(x)dx$.

$$\int x^3 dx = \frac{x^4}{4}$$

Verify this result by taking the derivative of $x^4/4$. We can establish a rule in this case, for which

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

This is the inverse of the first formula for derivatives in the table.

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Now, the **definite integral** can be found by the difference of the anti-derivatives at the limits of integration

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example 8:

Find the integral of $f(x) = x$ from 2 to 3.

$$\int_2^3 x dx = \frac{x^2}{2}\Big|_2^3 = \frac{1}{2}\left(3^2 - 2^2\right) = \frac{5}{2}$$

Property: Linearity

The anti-derivative and the integral are both linear operators, so

$$\int_a^b f(x) + cg(x)dx = \int_a^b f(x)dx + c\int_a^b g(x)dx \quad (7)$$

Technique: Integration by parts

We can integrate both sides of (3):

$$\begin{aligned}
\frac{d}{dx}g(x)h(x) &= g'(x)h(x) + g(x)h'(x) \\
\int \frac{d}{dx}g(x)h(x)dx &= \int g'(x)h(x) + g(x)h'(x)dx \\
g(x)h(x) &= \int g'(x)h(x)dx \\
&\quad + \int g(x)h'(x)dx
\end{aligned}$$

Re-arranging the terms yields

$$\int g(x)h'(x)dx = g(x)h(x) - \int g'(x)h(x)dx \quad (8)$$

This rule can be used when integrating the product of two functions, converting the left side of the equation into the right side. The choice of $g(x)$ and $h'(x)$ is critical and has to be done such that the integral on the right side is easier to compute.

Example 9:

Integrate $f(x) = \sin(x)x$. Note that when asked to integrate and the range has not been specified, it's equivalent to being asked to find the anti-derivative.

Let $f(x) = g(x)h'(x)$ where

$$g(x) = x \text{ and } h'(x) = \sin(x)$$

To construct the right hand side of (8) we need to find, $g'(x)$ and $h(x)$:

$$g'(x) = \frac{d}{dx}x = 1 \text{ and } h(x) = \int \sin(x)dx = -\cos(x)$$

Then we replace into (8) to obtain

$$\int \sin(x)xdx = -x\cos(x) + \int \cos(x)dx$$

The integral on the right hand side is certainly simpler than the original and we've already computed its answer in a previous example:

$$\int \sin(x)xdx = -x\cos(x) + \sin(x)$$

Try using $g(x) = \sin(x)$ and $h'(x) = x$ and verify that this only yields a more complex expression.

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You might need to use this method several times for a given integral until the remaining integral can be found directly or by other methods (e.g. $f(x) = \sin(x)x^2$). Another situation you might encounter is that the new integral is exactly the same as the original, for which you can factor the two together. Examples for these cases are not provided, but keep them in mind.

Technique: Variable substitution

This is better explained through an example. Consider the integral

$$\int \cos(x^2)xdx$$

Then consider the substitution $u = x^2$, where

$$\frac{du}{dx} = 2x \;\rightarrow\; \frac{du}{2} = xdx$$

The integral can now be written in terms of u and solved:

$$\int \cos(x^2)xdx = \frac{1}{2}\int \cos(u)du = \frac{1}{2}\sin(u)$$

The last step is to write the answer in term of x:

$$\frac{1}{2}\sin(u) = \frac{1}{2}\sin(x^2)$$

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Integration is generally more difficult that differentiation. There are no straight forward formulas and requires wise selection of $g(x)$ or $h'(x)$ to integrate by parts, and of u for variable substitution. Making the correct choices should become easy after some practice. Nevertheless, the methods and techniques shown here cover the basic techniques to solve any integral.